Chapter 4

Metric dimension of generalized Petersen graphs $P(n, 3)$

In this chapter, we study the metric dimension of the generalized Petersen graphs $P(n, 3)$ by giving a partial answer to an open problem raised in [24]: Is $P(n, m)$ for $n \geq 7$ and $3 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, a family of graphs with constant metric dimension? We prove that the generalized Petersen graphs $P(n, 3)$ have metric dimension equal to 3 for $n \equiv 1(\text{mod } 6)$, $n \geq 25$, and to 4 for $n \equiv 0(\text{mod } 6)$, $n \geq 24$. For the remaining cases only 4 vertices appropriately chosen suffice to resolve all the vertices of $P(n, 3)$, thus implying that $\dim(P(n, 3)) \leq 4$, except when $n \equiv 2(\text{mod } 6)$, when $\dim(P(n, 3)) \leq 5$.

4.1 Notation and auxiliary results

In [24] Javaid et al. proved that some regular graphs namely generalized Petersen graphs $P(n, 2)$, antiprisms $A_n$ and Harary graphs $H_{4,n}$ are families of graphs with constant metric dimension and raised an open problem.

Open Problem [24]: Is $P(n, m)$ for $n \geq 7$ and $3 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, a family of graphs with constant metric dimension?

In this chapter, we give a partial answer to this open problem and we show that
the generalized Petersen graphs $P(n, 3)$ constitute a family of regular graphs having bounded metric dimension and only 4 vertices appropriately chosen suffice to resolve all vertices of the generalized Petersen graphs $P(n, 3)$ except $n \equiv 2(\text{mod } 6)$, when this number equals 5. For $n \equiv 1(\text{mod } 6)$ a minimal resolving set has cardinality equal to 3.

In what follows all indices $i$ which do not satisfy inequalities $1 \leq i \leq n$ will be taken modulo $n$.

### 4.2 Upper bounds for metric dimension of generalized Petersen graphs $P(n, 3)$

The generalized Petersen graph denoted by $P(n, m)$, where $n \geq 3$ and $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, is a cubic graph having vertex set

$$V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$$

and edge set

$$E = \{u_iu_{i+1}, u_iv_i, v_iv_{i+m} : 1 \leq i \leq n\}.$$ 

Generalized Petersen graphs were first defined by Watkins [44]. For $m = 1$ the generalized Petersen graph $P(n, 1)$ is called prism, denoted by $D_n$. In [5] it was shown that

$$\dim(P_m \times C_n) = \begin{cases} 
2, & \text{if } n \text{ is odd;} \\
3, & \text{if } n \text{ is even.}
\end{cases}$$

Since the prism $D_n$ is actually the cross product of $P_2$ with a cycle $C_n$, this implies that
\[ dim(D_n) = \begin{cases} 
2, & \text{if } n \text{ is odd;} \\
3, & \text{if } n \text{ is even.} 
\end{cases} \]

So, prisms constitute a family of 3-regular graphs with bounded metric dimension. In [24] it was proved that \( \dim(P(n, 2)) = 3 \) for every \( n \geq 5 \). Now we will find the metric dimension of the generalized Petersen graphs \( P(n, 3) \) when \( n \equiv 0 \) or \( 1(\text{mod } 6) \) and an upper bound in the remaining cases.

When \( m = 3 \), \( \{u_1, u_2, ..., u_n\} \) induces a cycle in \( P(n, 3) \) with \( u_iu_{i+1} \) (\( 1 \leq i \leq n \)), as edges. If \( n = 3l \) (\( l \geq 3 \)), then \( \{v_1, v_2, ..., v_n\} \) induces 3 cycles of length \( l \), otherwise it induces a cycle of length \( n \) with \( v_iv_{i+3} \) (\( 1 \leq i \leq n \)), as edges. For example, \( P(8, 3) \) is the Möbius-Kantor graph [29].

Since generalized Petersen graphs \( P(n, 3) \) form an important class of 3-regular graphs with \( 2n \) vertices and \( 3n \) edges, it is desirable to find their metric dimensions. For our purpose, we call the cycle induced by \( \{u_1, u_2, ..., u_n\} \), outer cycle and cycle(s) induced by \( \{v_1, v_2, ..., v_n\} \) inner cycle(s). Note that the choice of appropriate basis vertices (also referred to as landmarks in [27]) is core of the problem.

**Theorem 4.2.1.** For the generalized Petersen graph \( P(n, 3) \) we have
(a) \( \dim(P(n, 3)) \leq 3 \) for \( n \equiv 1(\text{mod } 6) \) and \( n \geq 13 \);
(b) \( \dim(P(n, 3)) \leq 4 \) for \( n \equiv 0, 3, 4, 5(\text{mod } 6) \) and \( n \geq 17 \).
(c) \( \dim(P(n, 3)) \leq 5 \) for \( n \equiv 2(\text{mod } 6) \) and \( n \geq 8 \).

**Proof:** We denote \( W = \{v_1, v_2, v_3, u_4\} \) for \( n \equiv 0, 3, 4, 5(\text{mod } 6) \) and \( W = \{v_1, v_{3k-1}, v_{6k}\} \) for \( n \equiv 1(\text{mod } 6) \), \( n = 6k + 1 \). We show that the set \( W \) distinguishes the vertices of \( P(n, 3) \) for \( n \neq 2(\text{mod } 6) \). For this purpose, we give the representation of \( V(P(n, 3)) \) in these cases.

**Case (i)** \( n = 6k, k \in \mathbb{Z}^+, k \geq 2 \). For every \( n \geq 12 \), the representations of vertices on the outer cycle are
\[
r(u_1|W) = (1, 2, 3, 3), \quad r(u_2|W) = (2, 1, 2, 2), \quad r(u_3|W) = (3, 2, 1, 1). \]
\[ r(u_{2+3i} | W) = \begin{cases} 
(3, 2, 3, 1), & i = 1; \\
(i + 2, i + 1, i + 2), & 2 \leq i \leq k; \\
(2k - i + 2, 2k - i + 1, 2k - i + 2, 2k - i + 4), & k + 1 \leq i \leq 2k - 1. 
\end{cases} \]

\[ r(u_{3+3i} | W) = \begin{cases} 
(4, 3, 2, 2), & i = 1; \\
(i + 3, i + 2, i + 1, i + 3), & 2 \leq i \leq k - 1; \\
(2k - i + 1, 2k - i + 2, 2k - i + 1, 2k - i + 3), & k \leq i \leq 2k - 1. 
\end{cases} \]

\[ r(u_{4+3i} | W) = \begin{cases} 
(i + 2, i + 3, i + 2, i + 2), & 1 \leq i \leq k - 1; \\
(2k - i, 2k - i + 3, 2k - i + 2, 2k - i + 2), & k \leq i \leq 2k - 2. 
\end{cases} \]

Representations of vertices with respect to \( W \) on the inner cycles are

\[ r(v_{1+3i} | W) = \begin{cases} 
(i, i + 3, i + 2, i), & 1 \leq i \leq k; \\
(2k - i, 2k - i + 3, 2k - i + 4, 2k - i + 2), & k + 1 \leq i \leq 2k - 1. 
\end{cases} \]

\[ r(v_{2+3i} | W) = \begin{cases} 
(i + 3, i, i + 3, i + 1), & 1 \leq i \leq k; \\
(2k - i + 3, 2k - i, 2k - i + 3, 2k - i + 3), & k + 1 \leq i \leq 2k - 1. 
\end{cases} \]

\[ r(v_{3+3i} | W) = \begin{cases} 
(i + 4, i + 3, i, i + 2), & 1 \leq i \leq k - 1; \\
(2k - i + 2, 2k - i + 3, 2k - i, 2k - i + 2), & k \leq i \leq 2k - 1. 
\end{cases} \]
Case (ii) $n = 6k + 1, k \in \mathbb{Z}^+, k \geq 2$. For every $n \geq 13$, the representations of vertices with respect to $W$ are the following:

Representations of vertices on the outer cycle are

$$r(u_{3i}|W) = \begin{cases} 
(i + 2, k - i + 2, i + 2), & 1 \leq i \leq k - 1; \\
(k + 2, 2, k + 1), & i = k; \\
(2k - i + 3, i - k + 2, 2k - i + 1), & k + 1 \leq i \leq 2k - 1; \\
(3, k + 1, 1), & i = 2k.
\end{cases}$$

$$r(u_{3i-1}|W) = \begin{cases} 
(i + 1, k - i + 1, i + 1), & 1 \leq i \leq k; \\
(2k - i + 2, i - k + 1, 2k - i + 2), & k + 1 \leq i \leq 2k.
\end{cases}$$

$$r(u_{3i-2}|W) = \begin{cases} 
(i, k - i + 2, i + 2), & 1 \leq i \leq k; \\
(k + 1, 3, k + 2), & i = k + 1; \\
(2k - i + 3, i - k + 2, 2k - i + 3), & k + 2 \leq i \leq 2k; \\
(2, k + 2, 2), & i = 2k + 1.
\end{cases}$$

Representations of vertices with respect to $W$ on the inner cycle are

$$r(v_{3i-1}|W) = \begin{cases} 
(i + 2, k - i, i), & 1 \leq i \leq k - 1; \\
(k + 1, 0, k), & i = k; \\
(k, 1, k + 1), & i = k + 1; \\
(2k - i + 1, i - k, 2k - i + 3), & k + 2 \leq i \leq 2k.
\end{cases}$$

$$r(v_{3i}|W) = \begin{cases} 
(i + 3, k - i + 3, i + 3), & 1 \leq i \leq k - 2; \\
(i + 3, k - i + 3, 2k - i), & k - 1 \leq i \leq k; \\
(2k - i + 4, i - k + 3, 2k - i), & k + 1 \leq i \leq 2k - 2; \\
(5, k + 1, 1), & i = 2k - 1; \\
(4, k, 0), & i = 2k.
\end{cases}$$
Case (iii) \( n = 6k + 3, k \in \mathbb{Z}^+ \). For \( P(9, 3) \), the representations of the vertices are

\[ r(u_1|W) = (1, 2, 3, 3), \; r(u_2|W) = (2, 1, 2, 2), \; r(u_3|W) = (3, 2, 1, 1), \; r(u_5|W) = (3, 2, 3, 1), \; r(u_6|W) = (3, 3, 2, 2), \; r(u_7|W) = (2, 3, 3, 3), \]

\[ r(u_8|W) = (3, 2, 3, 4), \; r(u_9|W) = (2, 3, 2, 4), \; r(v_1|W) = (1, 4, 3, 1), \]

\[ r(v_5|W) = (4, 1, 4, 2), \; r(v_6|W) = (4, 4, 1, 3), \; r(v_7|W) = (1, 4, 4, 2), \]

\[ r(v_8|W) = (4, 1, 4, 3), \; r(v_9|W) = (3, 4, 1, 3). \]

For every \( n \geq 15 \), representations of vertices with respect to \( W \) are the following:

Representations of vertices on the outer cycle are

\[ r(u_1|W) = (1, 2, 3, 3), \; r(u_2|W) = (2, 1, 2, 2), \; r(u_3|W) = (3, 2, 1, 1). \]

\[ r(u_{2+3i}|W) = \]

\[
\begin{cases}
(3, 2, 3, 1), & i = 1; \\
(i + 2, i + 1, i + 2, i + 2), & 2 \leq i \leq k; \\
(k + 2, k + 1, k + 2, k + 3), & i = k + 1; \\
(2k - i + 3, 2k - i + 2, 2k - i + 3, 2k - i + 5), & k + 2 \leq i \leq 2k.
\end{cases}
\]
\[
\begin{align*}
\mathcal{R}(u_{3+3i}|W) &= \\
&\begin{cases}
(4, 3, 2, 2), & i = 1; \\
(i + 3, i + 2, i + 1, i + 3), & 2 \leq i \leq k - 1; \\
(k + 2, k + 2, k + 1, k + 3), & i = k; \\
(2k - i + 2, 2k - i + 3, 2k - i + 2, 2k - i + 4), & k + 1 \leq i \leq 2k.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mathcal{R}(u_{4+3i}|W) &= \\
&\begin{cases}
(i + 2, i + 3, i + 2, i + 2), & 1 \leq i \leq k - 1; \\
(k + 1, k + 2, k + 2, k + 2), & i = k; \\
(2k - i + 1, 2k - i + 2, 2k - i + 3, 2k - i + 3), & k + 1 \leq i \leq 2k - 1.
\end{cases}
\end{align*}
\]

Representations of the vertices with respect to \(W\) on the inner cycles are

\[
\begin{align*}
\mathcal{R}(v_{1+3i}|W) &= \\
&\begin{cases}
(i, i + 3, i + 2, i), & 1 \leq i \leq k; \\
(k, k + 3, k + 3, k + 1), & i = k + 1; \\
(2k - i + 1, 2k - i + 4, 2k - i + 5, 2k - i + 3), & k + 2 \leq i \leq 2k.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mathcal{R}(v_{2+3i}|W) &= \\
&\begin{cases}
(i + 3, i + 3, i + 1), & 1 \leq i \leq k; \\
(k + 3, k + 3, k + 2), & i = k + 1; \\
(2k - i + 4, 2k - i + 1, 2k - i + 4, 2k - i + 4), & k + 2 \leq i \leq 2k.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\mathcal{R}(v_{3+3i}|W) &= \\
&\begin{cases}
(i + 4, i + 3, i, i + 2), & 1 \leq i \leq k - 1; \\
(k + 3, k + 3, k, k + 2), & i = k; \\
(2k - i + 3, 2k - i + 4, 2k - i + 1, 2k - i + 3), & k + 1 \leq i \leq 2k.
\end{cases}
\end{align*}
\]
Case (iv) \( n = 6k + 4, k \in \mathbb{Z}^+ \). For every \( n \geq 10 \), the representations of the vertices on the outer cycle are
\[ r(u_1|W) = (1, 2, 3, 3), \quad r(u_2|W) = (2, 1, 2, 2), \quad r(u_3|W) = (3, 2, 1, 1). \]

\[ r(u_{2+3i}|W) = \begin{cases} 
(3, 2, 3, 1), & \text{i = 1}; \\
(i + 2, i + 1, i + 2, i + 2), & 2 \leq i \leq k; \\
(2k - i + 2, 2k - i + 3, 2k - i + 4, 2k - i + 4), & k + 1 \leq i \leq 2k. 
\end{cases} \]

\[ r(u_{3+3i}|W) = \begin{cases} 
(4, 3, 2, 2), & i = 1; \\
(i + 3, i + 2, i + 1, i + 3), & 2 \leq i \leq k; \\
(2k - i + 3, 2k - i + 2, 2k - i + 3, 2k - i + 5), & k + 1 \leq i \leq 2k. 
\end{cases} \]

\[ r(u_{4+3i}|W) = \begin{cases} 
(i + 2, i + 3, i + 2, i + 2), & 1 \leq i \leq k; \\
(2k - i + 2, 2k - i + 3, 2k - i + 2, 2k - i + 4), & k + 1 \leq i \leq 2k. 
\end{cases} \]

Representations of vertices on inner cycle are
\[ r(v_{1+3i}|W) = \begin{cases} 
(i, i + 3, i + 2, i), & 1 \leq i \leq k; \\
(k + 1, k + 4, k + 1, k + 1), & i = k + 1; \\
(2k - i + 4, 2k - i + 5, 2k - i + 2, 2k - i + 4), & k + 2 \leq i \leq 2k + 1. 
\end{cases} \]
Case (v) $n = 6k + 5, k \in \mathbb{Z}^+$. For every $n \geq 17$, the representations of vertices on the outer cycle are

$r(u_1|W) = (1, 2, 3, 3), r(u_2|W) = (2, 1, 2, 2), r(u_3|W) = (3, 2, 1, 1)$.
\( r(u_{3+3i}|W) = \)

\[
\begin{align*}
(4, 3, 2, 2), & \quad i = 1; \\
(i + 3, i + 2, i + 1, i + 3), & \quad 2 \leq i \leq k - 1; \\
(k + 2, k + 2, k + 1, k + 3), & \quad i = k; \\
(k + 1, k + 2, k + 2, k + 3), & \quad i = k + 1; \\
(2k - i + 2, 2k - i + 3, 2k - i + 4, 2k - i + 4), & \quad k + 2 \leq i \leq 2k.
\end{align*}
\]

\( r(u_{4+3i}|W) = \)

\[
\begin{align*}
(i + 2, i + 3, i + 2, i + 2), & \quad 1 \leq i \leq k - 1; \\
(k + 2, k + 2, k + 2, k + 2), & \quad i = k; \\
(k + 2, k + 1, k + 2, k + 3), & \quad i = k + 1; \\
(2k - i + 3, 2k - i + 2, 2k - i + 3, 2k - i + 5), & \quad k + 2 \leq i \leq 2k.
\end{align*}
\]

Representations of vertices with respect to \( W \) on inner cycle are

\( r(v_{1+3i}|W) = \)

\[
\begin{align*}
(i, i + 3, i + 2, i), & \quad 1 \leq i \leq k - 1; \\
(k, k + 2, k + 2, k), & \quad i = k; \\
(k + 1, k + 1, k + 3, k + 1), & \quad i = k + 1; \\
(k + 2, k, k + 3, k + 2), & \quad i = k + 2; \\
(2k - i + 5, 2k - i + 2, 2k - i + 5, 2k - i + 5), & \quad k + 3 \leq i \leq 2k + 1.
\end{align*}
\]
\[ r(v_{2+3i}|W) = \begin{cases} 
(i + 3, i, i + 3, i + 1), & 1 \leq i \leq k - 1; \\
(k + 3, k, k + 2, k + 1), & i = k; \\
(k + 3, k + 1, k + 1, k + 2), & i = k + 1; \\
(k + 2, k + 2, k, k + 2), & i = k + 2; \\
(2k - i + 4, 2k - i + 5, 2k - i + 2, 2k - i + 4), & k + 3 \leq i \leq 2k + 1. 
\end{cases} \]

\[ r(v_{3+3i}|W) = \begin{cases} 
(i + 4, i + 3, i, i + 2), & 1 \leq i \leq k - 2; \\
(k + 2, k + 2, k - 1, k + 1), & i = k - 1; \\
(k + 1, k + 3, k, k + 2), & i = k; \\
(k, k + 3, k + 1, k + 2), & i = k + 1; \\
(k - 1, k + 2, k + 2, k + 1), & i = k + 2; \\
(2k - i + 1, 2k - i + 4, 2k - i + 5, 2k - i + 3), & k + 3 \leq i \leq 2k. 
\end{cases} \]

We note that there are no two vertices in the inner cycle(s) with same representations.
Also, there are no two vertices in the inner cycle(s) and outer cycle having the same representations and no two vertices on outer cycle having the same representations.
This implies that \( W = \{v_1, v_2, v_3, u_4\} \) is a resolving set for \( V(P(n, 3)) \) when \( n \equiv 0, 3, 4, 5(\mod 6) \) implying that in these cases \( \text{dim}(P(n, 3)) \leq 4 \). Also \( W = \{v_1, v_{3k-1}, v_{6k}\} \) is a resolving set for \( n = 6k + 1 \), when \( \text{dim}(P(n, 3)) \leq 3 \).

**Case (vi) \( n = 6k + 2, k \in \mathbb{Z}^+ \).** It is straightforward to verify that \( W_1 = \{v_1, v_2, v_3, u_4\} \) and \( W_2 = \{v_1, v_2, u_7, u_{10}\} \) are resolving sets for \( P(8, 3) \) and \( P(14, 3) \), respectively. For every \( n \geq 20 \), we show that \( W = \{v_1, v_2, v_3, u_4, u_{3k+5}\} \) is a resolving set. For this, we first give representations of vertices with respect to \( W' = \{v_1, v_2, v_3, u_4\} \). The
representations of the vertices on the outer cycle are

\[ r(u_1|W') = (1, 2, 3, 3), \quad r(u_2|W') = (2, 1, 2, 2), \quad r(u_3|W') = (3, 2, 1, 1). \]

\[
r(u_{2+3i}|W') = \begin{cases} 
(3, 2, 3, 1), & i = 1; \\
(i + 2, i + 1, i + 2, i + 2), & 2 \leq i \leq k; \\
(2k - i + 2, 2k - i + 3, 2k - i + 2, 2k - i + 4), & k + 1 \leq i \leq 2k. 
\end{cases}
\]

\[
r(u_{3+3i}|W') = \begin{cases} 
(4, 3, 2, 2), & i = 1; \\
(i + 3, i + 2, i + 1, i + 3), & 2 \leq i \leq k - 1; \\
(k + 1, k + 2, k + 1, k + 3), & i = k; \\
(2k - i + 1, 2k - i + 2, 2k - i + 3, 2k - i + 3), & k + 1 \leq i \leq 2k - 1. 
\end{cases}
\]

\[
r(u_{4+3i}|W') = \begin{cases} 
(i + 2, i + 3, i + 2, i + 2), & 1 \leq i \leq k - 1; \\
(k + 2, k + 1, k + 2, k + 2), & i = k; \\
(2k - i + 2, 2k - i + 1, 2k - i + 2, 2k - i + 4), & k + 1 \leq i \leq 2k - 1. 
\end{cases}
\]

Representations of vertices with respect to \(W'\) on inner cycle are

\[
r(v_{1+3i}|W') = \begin{cases} 
(i, i + 3, i + 2, i), & 1 \leq i \leq k - 1; \\
(k, k + 1, k + 2, k), & i = k; \\
(k + 1, k, k + 3, k + 1), & i = k + 1; \\
(2k - i + 4, 2k - i + 1, 2k - i + 4, 2k - i + 4), & k + 2 \leq i \leq 2k. 
\end{cases}
\]
\( r(v_{2+3i}|W) = \)
\[
\begin{cases}
(i + 3, i, i + 3, i + 1), & 1 \leq i \leq k - 1; \\
(k + 3, k, k + 1, k + 1), & i = k; \\
(k + 2, k + 1, k, k + 2), & i = k + 1; \\
(2k - i + 3, 2k - i + 4, 2k - i + 1, 2k - i + 3), & k + 2 \leq i \leq 2k.
\end{cases}
\]

\( r(v_{3+3i}|W) = \)
\[
\begin{cases}
(i + 4, i + 3, i, i + 2), & 1 \leq i \leq k - 2; \\
(k + 1, k + 2, k - 1, k + 1), & i = k - 1; \\
(k, k + 3, k, k + 2), & i = k; \\
(k - 1, k + 2, k + 1, k + 1), & i = k + 1; \\
(2k - i, 2k - i + 3, 2k - i + 4, 2k - i + 2), & k + 2 \leq i \leq 2k - 1.
\end{cases}
\]

Consequently, \( r(u_{2+3k}|W) = r(u_{4+3k}|W) = (k + 2, k + 1, k + 2, k + 2); \)
\( r(u_{5+3k}|W) = r(u_{3+3k}|W) = (k + 1, k + 2, k + 1, k + 3); \)
\( r(u_{3k}|W) = r(v_{5+3k}|W) = (k + 2, k + 1, k, k + 2); \)
\( r(u_{8+3k}|W) = r(v_{3k}|W) = (k + 1, k + 2, k - 1, k + 1). \)

The vertex \( u_{3k+5} \) distinguishes these pairs of vertices with same representations as \( d(u_{3k+5}, u_{3k+2}) = 3, d(u_{3k+5}, u_{3k+4}) = 1, d(u_{3k+5}, u_{3k}) = 5, d(u_{3k+5}, v_{3k+5}) = 1, d(u_{3k+5}, v_{3k+8}) = 2, \) and \( d(u_{3k+5}, v_{3k}) = 4. \) This suggests that \( W = \{v_1, v_2, v_3, u_4, u_{3k+5}\} \) is a resolving set for \( V(P(n, 3)) \) in this case implying that \( \text{dim}(P(n, 3)) \leq 5. \)

### 4.3 Metric dimension of \( P(n, 3) \) for \( n \equiv 0, 1(\text{mod} \ 6) \)

In this section we will prove that \( \text{dim}(P(n, 3)) \geq 3 \) for \( n \equiv 1(\text{mod} \ 6) \) and \( n \geq 25 \) and \( \text{dim}(P(n, 3)) \geq 4 \) for \( n \equiv 0(\text{mod} \ 6) \) and \( n \geq 24, \) yielding exact values of \( \text{dim}(P(n, 3)) \) in these cases by Theorem 4.2.1. For this purpose we need some more notations and
definitions. Without loss of generality we can suppose that the vertices of the outer cycle are \( u_1, u_2, \ldots, u_n \) in the clockwise direction. For two vertices \( u_i \) and \( u_j \) \((i \neq j)\) we shall define the "clockwise distance" from \( u_i \) to \( u_j \), denoted by \( d^*(u_i, u_j) \) the distance, measured in clockwise direction, from \( u_i \) to \( u_j \), in the subgraph induced by the outer cycle. For example, \( d^*(u_1, u_n) = n - 1 \) and \( d^*(u_n, u_1) = 1 \); in general we have \( d^*(u_i, u_j) + d^*(u_j, u_i) = n \). This definition can be extended to any two vertices of \( P(n, 3) \) for \( i \neq j \) by:

\[
d^*(u_i, v_j) = d^*(v_i, u_j) = d^*(v_i, v_j) = d^*(u_i, u_j).
\]

Consider a vertex on the outer cycle, say \( u_1 \). A vertex \( u_i \) is called a good vertex for \( u_1 \) if \( u_i \) and \( u_{i+2} \) have equal distances to \( u_1 \), i.e., \( d(u_1, u_i) = d(u_1, u_{i+2}) \); otherwise \( u_i \) is called a bad vertex for \( u_1 \). This definition can be extended to vertices of the inner cycle: \( u_i \) is a good vertex for \( v_1 \) if \( d(v_1, u_i) = d(v_1, u_{i+2}) \) and bad otherwise.

In figure 4.1 we have represented by black dots all good vertices for \( u_1 \) when \( n = 6k + 1 \geq 25 \).

It is important to note that the set of good vertices for \( v_1 \) is deduced from the set of good vertices for \( u_1 \) by adding 4 new vertices, namely \( u_2, u_3, u_{6k-1} \) and \( u_{6k-2} \). Similarly, a vertex \( u_j \) is said to be good for the pair \( \{u_1, u_i\} \) if \( d(u_1, u_j) = d(u_1, u_{j+2}) \) and \( d(u_i, u_j) = d(u_i, u_{j+2}) \). If \( u_k \) is a good vertex for the pairs \( \{u_1, u_i\} \) and \( \{u_1, u_j\} \) then \( u_k \) is also a good vertex for the triplet \( \{u_1, u_i, u_j\} \), i.e., \( d(u_1, u_k) = d(u_1, u_{k+2}), d(u_i, u_k) = d(u_i, u_{k+2}) \) and \( d(u_j, u_k) = d(u_j, u_{k+2}) \).

Due to the rotational symmetry of \( P(n, 3) \) we deduce

**Lemma 4.3.1.** For any two vertices \( u_i \) and \( u_j \) on the outer cycle of \( P(n, 3) \) we have \( d(u_i, u_j) = d(u_{i+r}, u_{j+r}) \) for any \( 1 \leq r \leq n - 1 \).

In order to find good vertices for pairs of vertices belonging to the outer cycle the following lemma will be useful.

**Lemma 4.3.2.** Let \( 1 \leq i \leq n - 2 \). If \( u_j \) is good for \( u_1 \) and \( u_{j-i} \) is also good for \( u_1 \),
then \( u_j \) is good for the pair \( \{ u_1, u_{i+1} \} \).

**Proof:** By hypothesis we can write \( d(u_1, u_j) = d(u_1, u_{j+2}) \) and
\[ d(u_1, u_{j-i}) = d(u_1, u_{j-i+2}). \]
By Lemma 4.3.1 the last equality is equivalent to \( d(u_{i+1}, u_j) = d(u_{i+1}, u_{j+2}) \). \( \square \)

**Theorem 4.3.3.** \( \dim(P(n, 3)) = 3 \) if \( n = 6k + 1 \) and \( n \geq 25 \).

**Proof:** We shall prove that \( \dim(P(n, 3)) \geq 3 \) in this case, by showing that there is no resolving set of \( V(P(n, 3)) \) consisting of two vertices, \( X \) and \( Y \). If both vertices \( X \) and \( Y \) belong to the outer cycle, we can suppose that \( X = u_1 \). We distinguish three cases:

1) \( d^*(u_1, Y) \equiv 0 \text{ (mod 3)} \). We choose vertex \( u_{6k-4} \). Since vertices \( u_{6k-7}, u_{6k-10}, \ldots, u_{3k+5}, u_{3k+2}, u_{3k-1}, u_{3k-4}, \ldots, u_8, u_5 \) are good vertices for \( u_1 \), but \( u_2, u_{6k} \)
and \( u_{6k-3} \) are bad vertices for \( u_1 \) (see fig. 4.1), applying Lemma 4.3.2 we find that \( u_{6k-4} \) is a good vertex for any pair \( \{u_1, Y\} \) such that \( Y \notin \{u_{6k-5}, u_{6k-2}, u_{6k+1}\} \).

But in this case \( u_5 \) is a good vertex for the pairs \( \{u_1, u_{6k-2}\} \) and \( \{u_1, u_{6k+1}\} \) if \( k \geq 4 \) and for \( \{u_1, u_{6k-5}\} \) if \( k \geq 5 \); for \( k = 4 \) we have that \( u_{3k} = u_{12} \) is not a good vertex for \( u_1 \). It remains to consider the pair \( \{u_1, u_{6k-5}\} = \{u_1, u_{19}\} \) when \( k = 4 \). In this case \( u_9 \) is a good vertex for this pair. It follows that any pair \( \{u_1, Y\} \) cannot be a resolving set having two vertices.

2) \( d^*(u_1, Y) \equiv 1(\text{mod } 3) \). We consider vertex \( u_5 \). By starting from \( u_5 \) and going in the counter-clockwise direction at distances 1,4,7,... the only bad vertices encountered are \( u_4, u_1 \) and \( u_{6k-1} \). By Lemma 4.3.2 we deduce that \( u_5 \) is a good vertex for any pair \( \{u_1, Y\} \) such that \( Y \notin \{u_2, u_5, u_8\} \). In a similar manner we get that \( u_{6k-4} \) is a good vertex for the pairs \( \{u_1, u_2\} \) and \( \{u_1, u_5\} \) if \( k \geq 4 \) and for \( \{u_1, u_8\} \) if \( k \geq 5 \). For \( k = 4 \) the vertex \( u_{16} \) is a good vertex for \( \{u_1, u_5\} \).

3) \( d^*(u_1, Y) \equiv 2(\text{mod } 3) \). In this case in order to minimize the number of bad vertices for the pairs \( \{u_1, Y\} \) we choose vertex \( u_6 \). This vertex is a good vertex for any pair \( \{u_1, Y\} \) such that \( Y \notin \{u_3, u_6, u_9\} \). Vertex \( u_{6k-5} \) is a good vertex for all pairs \( \{u_1, u_3\}, \{u_1, u_6\} \) and \( \{u_1, u_9\} \) for any \( k \geq 4 \).

If both \( X \) and \( Y \) belong to the inner cycle we can consider that \( X = v_1 \) and \( Y = v_i \) \((i > 1)\); this case can be reduced to the case when \( X = u_1 \) and \( Y = u_i \) since the set of good vertices for \( v_p \) includes the set of good vertices for \( u_p \) for every \( 1 \leq p \leq n \).

If \( X = u_i \) and \( Y = v_i \) then any good vertex for \( u_i \) is also a good vertex for \( v_i \), hence for the pair \( \{X, Y\} \). The remaining case when \( X = u_i, Y = v_j \) and \( i \neq j \) can also be reduced to the case \( X = u_i, Y = u_j \). It follows that there is no resolving set containing two vertices, which concludes the proof. \( \square \)

**Theorem 4.3.4.** If \( n = 6k \) and \( n \geq 24 \) then \( \dim(P(n, 3)) = 4 \).
Proof: By Theorem 4.2.1 it is necessary only to show that \( \dim(P(n,3)) \geq 4 \), or that there is no resolving set of \( V(P(n,3)) \) consisting of three vertices, \( X, Y \) and \( Z \). By the same reasoning as in the proof of Theorem 4.3.1 it is sufficient to consider only the case when \( X, Y, Z \) belong to the outer cycle since the set of good vertices for \( v_1 \) can be deduced from the set of good vertices for \( u_1 \) (represented in figure 4.2) by adding vertices \( u_2, u_3, u_{6k-2} \) and \( u_{6k-3} \). As in the case \( n = 6k + 1 \) we shall see that for any three vertices \( X, Y, Z \) such that \( d^*(X,Y) < d^*(X,Z) \) it is possible to find a pair of vertices at distance 2 on the outer cycle, \( \{u_i, u_{i+2}\} \) having equal distances to \( X, Y \) and \( Z \), respectively. If \( n = 6k \) and \( X, Y, Z \) are on the outer cycle, we can suppose that \( X = u_1 \). By denoting \( (x, y) \equiv (a, b)(\text{mod } 3) \) if \( x \equiv a(\text{mod } 3) \) and \( y \equiv b(\text{mod } 3) \), the following 9 cases can occur: \( (d^*(u_1, Y), d^*(u_1, Z)) \) is congruent modulo 3 to:

1)\((0,0)\); 2)\((1,1)\); 3)\((2,2)\); 4)\((0,1)\); 5)\((0,2)\); 6)\((1,0)\); 7)\((1,2)\); 8)\((2,0)\); 9)\((2,1)\).

Some of these cases can be reduced to another cases. For example, from case 2 by permutation \( X \to Y, Y \to Z, Z \to X \) we obtain case 5 and by permutation \( X \to Z, Y \to X, Z \to Y \) we get case 8.

The graph of reducibility between cases is illustrated in figure 4.3.

It follows that it is sufficient to consider only cases 1,2,3,7,9.

Case 1. If we choose good vertex \( u_6 \) and we go in the counter-clockwise direction, reaching vertices \( u_3, u_{6k}, u_{6k-6}, \ldots, u_9 \) we encounter only two bad vertices, \( u_3 \) and \( u_{6k-3} \) (see figure 4.2). By Lemma 3.2 it follows that \( u_6 \) is a good vertex for all pairs \( \{u_1, Y\} \) where \( d^*(u_1, Y) \equiv 0(\text{mod } 3) \) and \( Y \notin \{u_4, u_{10}\} \). This implies that \( u_6 \) is a good vertex for all triplets \( \{u_1, Y, Z\} \), unless \( Y = u_4 \) and \( Z \in \{u_7, u_{10}, u_{13}, \ldots, u_{6k-2}\} \); \( Y = u_{10} \) and \( Z \in \{u_7, u_{13}, u_{16}, \ldots, u_{6k-2}\} \). For these triplets we must find other good
vertices on the outer cycle. Similarly, $u_{3k+6}$ is a good vertex for $u_1$ since $6k - 6 \geq 3k + 6$ and for all pairs $\{u_1, Y\}$ where $d^*(u_1, Y) \equiv 0 \pmod{3}$ and $Y \notin \{u_{3k+4}, u_{3k+10}\}$. Consequently, we have found a good vertex ($u_6$ or $u_{3k+6}$) for all triplets $\{u_1, Y, Z\}$ such that $\{Y, Z\} \neq \{u_4, u_{3k+4}\}, \{u_4, u_{3k+10}\},$ $\{u_{10}, u_{3k+4}\}, \{u_{10}, u_{3k+10}\}$. Finally, $u_9$ is a good vertex for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 1 \pmod{3}$ and $Y \notin \{u_7, u_{13}\}$. Since $k \geq 4$ we have $3k + 4 > 13$ and $u_9$ is a good vertex for the remaining triplets $\{u_1, Y, Z\}$, where $Y \in \{u_4, u_{10}\}$ and $Z \in \{u_{3k+4}, u_{3k+10}\}$.

Case 2. In a similar way we get that $u_{6k-5}$ is a good vertex for all pairs $\{u_1, Y\}$, where $d^*(u_1, Y) \equiv 1 \pmod{3}$ and $Y \notin \{u_{6k-7}, u_{6k-1}\}$, therefore $u_{6k-5}$ is a good vertex for all triplets $\{u_1, Y, Z\}$, unless $Y = u_{6k-7}$ and $Z \in \{u_2, u_5, \ldots, u_{6k-10}, u_{6k-4}, u_{6k-1}\}$.

Figure 4.2: Good vertices for $u_1$ ($n = 6k$)
Figure 4.3: Reducibility between cases

\[ Y = u_{6k-1} \text{ and } Z \in \{u_2, u_5, \ldots, u_{6k-10}, u_{6k-4}\} \]. Since \( 6k - 8 > 3k - 2 \) it follows that \( u_{6k-8} \) is a good vertex for \( u_1 \) and for all pairs \( \{u_1, Y\} \), where \( d^*(u_1, Y) \equiv 1(\text{mod } 3) \) and \( Y \notin \{u_{6k-10}, u_{6k-4}\} \). We have found a good vertex (\( u_{6k-5} \) or \( u_{6k-8} \)) for all triplets \( \{u_1, Y, Z\} \) such that \( \{Y, Z\} \neq \{u_{6k-7}, u_{6k-10}\}, \{u_{6k-7}, u_{6k-4}\}, \{u_{6k-1}, u_{6k-10}\}, \{u_{6k-1}, u_{6k-4}\} \).

Since \( k \geq 4 \) we find for triplets \( \{u_1, u_{6k-7}, u_{6k-10}\}, \{u_1, u_{6k-7}, u_{6k-4}\}, \{u_1, u_{6k-1}, u_{6k-10}\} \) and \( \{u_1, u_{6k-1}, u_{6k-4}\} \) good vertices \( u_{3k-4} \), \( u_{3k-4} \), \( u_{6k-11} \) and \( u_{3k-1} \), respectively (e. g., using Lemma 4.3.2).

**Case 3.** We deduce that \( u_5 \) is a good vertex for all pairs \( \{u_1, Y\} \), where \( d^*(u_1, Y) \equiv 2(\text{mod } 3) \) and \( Y \notin \{u_3, u_9\} \). It follows that \( u_5 \) is a good vertex for all triplets \( \{u_1, Y, Z\} \), unless \( Y = u_3 \) and \( Z \in \{u_6, u_9, \ldots, u_{6k}\}; Y = u_9 \) and \( Z \in \{u_6, u_{12}, u_{15}, \ldots, u_{6k}\} \). Also \( u_{3k-1} \) is a good vertex for \( u_1 \) and for all pairs \( \{u_1, Y\} \), where \( d^*(u_1, Y) \equiv 2(\text{mod } 3) \) and \( Y \notin \{u_{3k-3}, u_{3k+3}\} \). It follows that there exists a good vertex (\( u_5 \) or \( u_{3k-1} \)) for all triplets \( \{u_1, Y, Z\} \) such that \( \{Y, Z\} \neq \{u_3, u_{3k-3}\}, \{u_3, u_{3k+3}\}, \{u_9, u_{3k-3}\} \),
\{u_9, u_{3k+3}\} (note that for \(k = 4\) the third triplet must be eliminated from the list).

Now \(u_8\) is a good vertex for all pairs \(\{u_1, Y\}\), where \(d^*(u_1, Y) \equiv 2\) (mod 3) and \(Y \notin \{u_6, u_{12}\}\). It follows that \(u_8\) is a good vertex for all four remaining triplets, except \(\{u_1, u_3, u_{12}\}\) for \(k = 5\) and \(n = 30\). For this last triplet \(u_{3k+1} = u_{16}\) is a good vertex.

Cases 7 and 9 can be reduced to the case 7 without imposing any inequality between the distances \(d^*(X, Y)\) and \(d^*(X, Z)\).

Let \(A = \{u_2, u_5, u_8, \ldots, u_{6k-1}\}\) and \(B = \{u_3, u_6, u_9, \ldots, u_{6k}\}\). It is necessary to prove that for any triplet \(\{u_1, Y, Z\}\), where \(Y \in A\) and \(Z \in B\) there is a good vertex on the outer cycle. From the previous case we have seen that \(u_5\) is a bad vertex for pairs \(\{u_1, Z\}\), where \(Z \in B\) if and only if \(Z \in \{u_3, u_9\}\) and a bad vertex for pairs \(\{u_1, Y\}\), where \(Y \in A\) if and only if \(Y \in \{u_2, u_5, u_8, u_{3k+8}, u_{3k+11}, \ldots, u_{6k-1}\}\). It follows that \(u_5\) is a good vertex for all triplets \(\{u_1, Y, Z\}\), where \(Y \in A\) and \(Z \in B\), unless: \(Y \in A\) and \(Z \in \{u_3, u_9\}\); \(Y \in \{u_2, u_5, u_8, u_{3k+8}, u_{3k+11}, \ldots, u_{6k-1}\}\) and \(Z \in B\) (these sets of pairs \(\{Y, Z\}\) will be denoted by \(\alpha\) and \(\beta\), respectively). For the remaining triplets \(\{u_1, Y, Z\}\), where \(\{Y, Z\} \in \alpha \cup \beta\) we must find other good vertices on the outer cycle.

Consider now vertex \(u_{3k+1}\). This vertex is a good vertex for all pairs \(\{u_1, Y\}\), where \(Y \in A \setminus \{u_{3k-1}, u_{3k+5}\}\). Since \(3k - 7 \geq 5\) it follows that \(u_{3k+1}\) is also a good vertex for all pairs \(\{u_1, Z\}\), where \(Z \in \{u_3, u_9\}\). Therefore the set \(\alpha\) is reduced to the set \(\alpha_1\) of pairs \(\{Y, Z\}\) such that \(Y \in \{u_{3k-1}, u_{3k+5}\}\) and \(Z \in \{u_3, u_9\}\). Now \(u_8\) is a good vertex for all pairs \(\{u_1, Y\}\), where \(Y \in \{u_{14}, u_{17}, \ldots, u_{3k+8}\}\) and all pairs \(\{u_1, Z\}\), where \(Z \in \{u_3, u_9, u_{15}, u_{18}, \ldots, u_{6k}\}\). Since \(3k - 1 \geq 14\) for \(k \geq 5\) it follows that \(u_8\) is a good vertex for all pairs in \(\alpha_1\) if \(k \geq 5\) and for \(k = 4\) we must consider only the pairs \(\{u_{11}, u_3\}\) and \(\{u_{11}, u_9\}\). From figure 2 we deduce that \(u_{3k+3} = u_{15}\) is a good
vertex for \( \{u_1, u_{11}, u_3\} \) and \( u_{3k+4} = u_{16} \) is a good vertex for \( \{u_1, u_{11}, u_9\} \).

It remains to find good vertices for pairs in \( \beta \). Since \( u_{6k-5} \) is a good vertex for all pairs \( \{u_1, Y\} \), where \( Y \in A \setminus \{u_{6k-7}, u_{6k-1}\} \) and a good vertex for all pairs \( \{u_1, Z\} \), where \( Z \in \{u_{3k-3}, u_{3k}, \ldots, u_{6k-9}\} \), \( \beta \) is reduced to \( \gamma \cup \delta \), where \( \gamma \) consists of all pairs \( \{Y, Z\} \) with \( Y \in \{u_{6k-7}, u_{6k-1}\} \) and \( Z \in B \) and \( \delta \) of \( \{Y, Z\} \) with \( Y \in \{u_2, u_5, u_8, u_{3k+8}, u_{3k+11}, \ldots, u_{6k-10}, u_{6k-4}\} \) and \( Z \in \{u_3, u_6, \ldots, u_{3k-6}, u_{6k-6}, u_{6k-3}, u_{6k}\} \).

\( u_{3k-1} \) is a good vertex for all pairs \( \{u_1, Y\} \), where \( Y \in \{u_{3k+5}, u_{3k+8}, \ldots, u_{6k-1}\} \) and all pairs \( \{u_1, Z\} \), where \( Z \in B \setminus \{u_{3k-3}, u_{3k+3}\} \). Therefore \( \gamma \) is reduced to \( \varepsilon \), consisting of all pairs \( \{Y, Z\} \) such that \( Y \in \{u_{6k-7}, u_{6k-1}\} \) and \( Z \in \{u_{3k-3}, u_{3k+3}\} \) and \( \delta \) is reduced to \( \mu \), which consists of \( \{Y, Z\} \) with \( Y \in \{u_2, u_5, u_8\} \) and \( Z \in \{u_3, u_6, u_{3k-6}, u_{6k-6}, u_{6k-3}, u_{6k}\} \).

\( u_6 \) is a good vertex for all pairs \( \{u_1, Y\} \), where \( Y \in \{u_2, u_{3k+8}, u_{3k+11}, \ldots, u_{6k-1}\} \) and all pairs \( \{u_1, Z\} \), where \( Z \in \{u_{12}, u_{15}, \ldots, u_{3k+6}\} \).

If \( k \geq 5 \) then \( 6k-7 \geq 3k+8 \) and \( 3k-3 \geq 12 \); therefore \( u_6 \) is a good vertex for all triples \( \{u_1, Y, Z\} \), where \( \{Y, Z\} \in \varepsilon \). If \( k = 4 \) the pairs in \( \varepsilon \) are \( \{u_{17}, u_9\}, \{u_{17}, u_{15}\}, \{u_{23}, u_9\} \) and \( \{u_{23}, u_{15}\} \); good vertices for these pairs are vertices \( u_{24}, u_5, u_{16} \) and \( u_9 \), respectively, which also are good vertices for \( u_1 \).

It remains to study the pairs from \( \mu \). \( u_{3k} \) is a good vertex for all pairs \( \{u_1, Y\} \), where \( Y \in \{u_2, u_5, \ldots, u_{3k-4}\} \) and all pairs \( \{u_1, Z\} \), where \( Z \in \{u_{3k+6}, u_{3k+9}, \ldots, u_{6k}\} \). Since \( 3k-4 \geq 8 \) and \( 6k-6 \geq 3k+6 \) it follows that \( \mu \) is reduced to the set \( \nu \) of pairs \( \{Y, Z\} \), where \( Y \in \{u_2, u_5, u_8\} \) and \( Z \in \{u_3, u_6, u_{3k-6}\} \).

Finally, \( u_{3k+1} \) is a good vertex for all pairs \( \{u_1, Y\} \), where \( Y \in A \setminus \{u_{3k-1}, u_{3k+5}\} \) and for all pairs \( \{u_1, Z\} \), where \( Z \in \{u_3, u_6, \ldots, u_{3k-3}\} \).
Consequently, $u_{3k+1}$ is a good vertex for $u_1$ and all pairs from $\nu$. \qed